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REVIEWS

Geometrical Methods of Mathematical Physics. By B. F. Schutz. Cambridge University Press, 1980. 250 pp. £20 (hardback), £7.95 (paperback).

Fluid dynamics remains perhaps the one major branch of theoretical physics in which differential geometry has made no significant impact; the major problems are thought to lie in analysis, the geometry being a subsidiary consideration. However, as first realized by Cartan, geometry and analysis are intimately interwoven, and this fact has dominated modern relativity and quantum physics. What does it offer for fluid dynamics?

Schutz has tried to provide an elementary text on modern differential geometry for the applied mathematician. Rather wisely he has concentrated on three main topics: differentiable manifolds and tensor calculus, Lie derivatives and groups, and differential forms. As an introduction these three chapters are excellent. It would require a great deal of delving in the literature to produce equivalent treatments. The treatment of Lie derivatives is particularly good. (However, readers of JFM may wish to be reminded that at a slightly higher level of sophistication there are two admirable texts. *Analysis, Manifolds and Physics* by Choquet-Bruhat, de Witt-Morette & Dillard-Bleick (North-Holland, 1977) is an encyclopaedic mathematically oriented text which explores the geometry–analysis interface thoroughly. *Mathematical Methods of Classical Mechanics* by Arnol'd (Springer Verlag, 1978) does the same but in a more intuitive physically oriented manner. Anyone who understands Hamiltonian mechanics should have no difficulty in picking up the concepts from this book. Inexplicably Arnol'd's admirable book is missing from Schutz's bibliography.

Unfortunately the potentially most stimulating chapter in the book on physical applications citing examples from thermodynamics, Hamiltonian mechanics, electromagnetism, fluid dynamics and relativity is perhaps too superficial. The sceptical hydrodynamicist who skims this chapter first is not likely to find much to make him read the rest of the book. Because of this there follow here some brief notes on conservation of circulation, vorticity and helicity in the notation of Schutz, based on an original idea of Brandon Carter.

Latin letters denote vectors, Greek letters forms, d the exterior derivative, $\wedge$ the exterior product, $v.\omega$ denotes the contraction of a vector v and p-form ω across adjacent indices, $\mathscr{L}_v$ is the Lie derivative with respect to the vector field v. The fundamental results are:

(i) (Poincaré) $dd\omega = 0 \quad \forall p$-forms ω;

(ii) (Cartan) $\mathscr{L}_v\, \omega = v.d\omega + d(v.\omega) \quad \forall$ vectors v, p-forms ω;

(iii) (Stokes) $\int_S d\omega\, dS = \int_{\partial S} \omega\, dS$;

where ω is a p-form, S a $(p+1)$-dimensional volume, ∂S the p-dimensional boundary of S, and dS the appropriate volume element.

For a perfect fluid with 3-velocity $\mathbf{v}$, there is a 4-velocity $v^a = (\mathbf{v}, 1)\,(a = 1, 2, 3, 4)$, a Lagrangian $L = L(x^a, v^a)$, an effective momentum $\pi_a = \partial L/\partial v^a$, and the Euler equations can be written as

$$\mathscr{L}_v \pi = dL. \qquad (1)$$

There is also a Hamiltonian formalism: let $\omega = d\pi$ be the vorticity 2-form and define $H = v.\pi - L$. Using the Cartan formula and (1) it follows immediately that

$$dH = -v.\omega. \tag{2}$$

Let C be any closed (1-dimensional) curve in spacetime. The *circulation* around C is

$$\mathscr{C}(C) = \oint_C \pi\, dS,$$

where dS is here a 1-dimensional line element. If C comoves with the fluid,

$$\begin{aligned}\mathscr{L}_v(\mathscr{C}(C)) &= \oint_C \mathscr{L}_v \pi\, dS \\ &= \oint_C dL\, dS = [L]_C = 0,\end{aligned}$$

since C is closed. We have conservation of circulation. Next let S be a comoving 2-dimensional surface in spacetime. The *vorticity flux* across S is

$$W(S) = \oint_S \omega\, dS = \oint_S d\pi\, dS = \oint_{\partial S} \pi\, dS = C(\partial S),$$

where Stokes' theorem has been used. Thus a comoving surface element conserves its vorticity flux. Alternatively one can derive the differential equation form via

$$\begin{aligned}\mathscr{L}_v w &= v.d\omega + d(v.\omega) \quad \text{(Cartan)} \\ &= v.(dd\pi) - d(dH) \quad \text{using (2)} \\ &= 0 \quad \text{(Poincaré)}. \end{aligned} \tag{3}$$

Given a comoving volume or 3-surface Σ, there is only one non-trivial 3-form $\pi \wedge \omega$, and the *helicity integral* is

$$\mathscr{H}(\Sigma) = \int_\Sigma \pi \wedge \omega\, dS.$$

Then

$$\begin{aligned}\mathscr{L}_v \mathscr{H}(\Sigma) &= \int_\Sigma [(\mathscr{L}_v \pi) \wedge \omega + \pi \wedge \mathscr{L}_v \omega]\, dS \\ &= \int_\Sigma dL \wedge \omega\, dS \quad \text{using (1), (3)} \\ &= \int_\Sigma d(L\omega)\, dS \quad \text{using (3)} \\ &= \int_{\partial\Sigma} L\omega\, dS.\end{aligned}$$

Now suppose that Σ is surrounded by a region of irrotational flow. Then the helicity of Σ is conserved. There are essentially no other 3-forms. The only non-trivial 4-form is $\omega \wedge \omega$. For isentropic flow $dH = 0$ and from (2) it follows that the rank of ω (necessarily even) is less than 4. Thus rank $(\omega) \leqslant 2$, which implies $\omega \wedge \omega = 0$. Thus there are no other conserved integrals of this type.

The last chapter of the book describes affine connections with applications to relativity and gauge theories, and could usefully be enlarged.

As the above example illustrates, differential-geometric methods have their place in fluid mechanics. Schutz's book forms a very useful introduction to the more advanced texts cited. However it is definitely not oriented towards fluid dynamics.

J. M. STEWART

The Fluid Mechanics of Large Blood Vessels. By T. J. PEDLEY. Cambridge University Press, 1980. 446 pp. £35.

This is a new type of book in the area of biomechanics and it signals a coming of age for the field. Most previous works, aimed principally at biological or medical audiences, omit mathematical development. Dr Pedley's book concentrates on the mathematical analysis of the fluid mechanics of interest in the large blood vessels. It is in this aspect that the general tone of the book sets a new level: it is tuned to the analytical requirements for the discussion of the subject rather than to a general audience. It is a book in which medical students may find difficulty with some of the mathematical analysis, but which will be informative for students of fluid mechanics and applied mathematics.

At the same time Dr Pedley has kept his feet on the ground in clearly and concisely summarizing the experimental and biological background of the various topics treated mathematically. Chapter 1 is a physiological introduction of some seventy pages which should serve very well for the inexperienced as well as the experienced workers in blood flow mechanics. In the preface some historical notes are given and a quotation of Thomas Young, a physician for whom mechanics and biology were intertwined. As Young put it in 1809, the circulation of the blood 'must become simply a question belonging to the most refined departments of the theory of hydraulics'. Dr Pedley's book exemplifies the application of refined theory to blood flow.

Chapter 2 discusses the theory of pressure pulse propagation in arteries and treats both linear and nonlinear aspects of the theory. A concise and clear summary is given of the one dimensional theories which have become usual for this kind of study. The two-dimensional theory given shows the principal effects of interest for typical arterial flows including the effect of longitudinal tethering. Effects of tapering, branching, and wave reflections are also discussed. The topic of wave propagation in the arteries has by now such a large literature that many aspects that might have been included are omitted, such as the effect of bending stresses in the wall or the compressibility of the blood. These and other refinements give interesting physical effects but are probably not important in arterial blood flow. Dr Pedley has given the bulk of what is useful.

Chapters 3, 4 and 5 concerning details of flow patterns and wall shear stress distribution in arteries comprise the major portion of the book and for these topics there is no other connected account in print. Chapter 3 covers the shear stress distribution in straight tubes, including entry flow and unsteady and reversing flows. The opening section of this chapter is an interesting essay on the difficulty of measuring wall shear stress experimentally.

Chapter 4 is concerned with the flow patterns and shear stresses in curved tubes. It is a long and thorough chapter on the subject and anyone interested in flow in curved tubes will enjoy it, whether they are interested in blood flow or not. Chapter 5 deals with flow patterns and wall shear stresses in branched tubes. This is a topic in which experimental and theoretical information on details of flow patterns and shear stresses have only recently been developed. At the rate at which current research work is proceeding, it is a chapter that will probably deserve rewriting in a comparatively short time. A very interesting closing section of the chapter deals with instability of flow in the aorta. Something like short bursts of turbulence are regularly seen shortly after the time of peak velocity in the aorta of man and other large mammals.

The final Chapter 6 deals with collapsible tubes which is applicable to venous flow and in some cases to arterial flow, particularly the mechanism of the Korotkoff sounds which are used as indicators in the measurement of arterial blood pressure. The introduction to this section covering the physiological and experimental background is an excellent summary of the literature to date. The subsequent analyses given are largely drawn from the work of Dr Pedley and his associates. The development of a lump parameter model to explain oscillations observed is a good example of obtaining a maximum amount of information analytically from a relatively simple mathematical model. This chapter also represents a field currently under rapid development so that it will be worthwhile to rewrite or extend this chapter in a relatively short time, if not already.

The book contains an extensive appendix on the analysis of a hot film anemometer. Almost all velocity profiles measured in models or *in vivo* have used this type of probe, so the study of the device is appropriate to the subject. The analyses themselves will be of interest to workers in fluid mechanics who will use this or similar devices. It will probably not be easily read by people who would like to use the device but are not trained analytically. The conclusions as to the dangers of using the hot-film anemometer in unsteady and reversing flows should be readily appreciated.

This book is an important contribution to the literature of fluid mechanics in general and biofluid mechanics in particular. It may be subject to criticism by biologists and medical people who will feel that the analytical results are not easily applied or utilized for particular purposes or pathological situations. It probably would have been useful to give some specific advice where possible in the form of summaries of the theory for a potential user and a discussion of the situations to which the theoretical results apply. This is largely done in introductory paragraphs but it might be useful to have summaries after the theory also. Such criticisms will probably be outweighed by the appreciation of those who are able to follow the analysis for an authoritative and up-to-date account of the fluid mechanics of large blood vessels which adequately explains the mathematical methods involved.

As mentioned at the outset, the publication of a book of this type indicates that biomechanics is coming of age. It is abundantly clear that sophisticated analytical techniques are useful in understanding biomechanical phenomena of medical interest. Now it is time to turn to practical applications and integration with other fields. For large blood vessels, this means such things as optimization of counterpulsation devices used in the large blood vessels, the study of the entire history, cellular response, and mass transport in atherosclerosis as integrated with the fluid mechanics, the control of blood pressure and hypertension, the growth response of arterial walls to stress, and integrated treatment of the large blood vessels and the mechanics of the microcirculation. These are all topics not covered in Dr Pedley's fine book and are mentioned to indicate that a good deal more research related to fluid mechanics would be useful.

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